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DESIGN OF REINFORCED PLATES UNDER CONSTRAINTS ON STRENGTH PROPERTIES

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The stress-strain state of a shell reinforced by high-modulus fibers was analyzed by Annin et al. [1], Kalamkarov and Kolpakov [2], and Annin et al. [3] using the asymptotic method of averaging in the theory of elastic shells combined with the approximate method of solving local problems of the averaging theory [4]. In the present paper, the results obtained are applied to the solution of design problems.

1. Calculation of Reinforced Plates. We first present the previous results [1–4] used in studying the elastic problem for a thin composite layer whose characteristic thickness is $\varepsilon \ll 1$. In [1–4], the composite formed by the laminates of parallel fibers (see Fig. 1) was considered. The fibers in the layers are at the same distance δ_f, and the layers are δ_b distant from one another.

The limiting (when $\varepsilon \to 0$) mechanical characteristics of the plate are calculated based on the solution of local problems in a cell of a periodic-structure plate [1–4]. In the case considered, for an approximate solution of local problems, one can use the model of rigid fibers in a soft matrix, which was proposed by the author [4] based on the following assumptions:

(1) the influence of the matrix on displacements of the fiber frames can be ignored;

(2) displacements of the matrix are determined from the solution of the elastic problem under the condition of ideal cohesion at the fiber–matrix interface.

The exact solution of the problem in Sec. 1 was derived by Kalamkarov [5]. The problem of Sec. 2 can be solved only by numerical methods. However, it is possible to estimate matrix strains, as was done in [1–4].

Rigidities of Reinforced Plates. In accordance with the model of rigid fibers in a soft matrix, the tensile rigidity S^0_{ijkl}, the skew-symmetric part of the rigidities S^1_{ijkl}, and the flexural rigidities S^2_{ijkl} of a plate are of the form [5]

$$S^0_{ijkl}(\mathbf{u}) = E_f \sum_{\beta=1}^{N} b_{ijkl}(\varphi_\beta)\gamma, \qquad S^1_{ijkl}(\mathbf{u}) = E_f \sum_{\beta=1}^{N} b_{ijkl}(\varphi_\beta) a_\beta \gamma,$$
$$S^2_{ijkl}(\mathbf{u}) = E_f \sum_{\beta=1}^{N} b_{ijkl}(\varphi_\beta) c_{ijkl}(\varphi_\beta)\gamma, \tag{1.1}$$

where $\gamma = \pi R^2/[(2R+\delta_f)(2R+\delta_b)]$; i, j, k, $l = 1$ and 2; $\mathbf{u}$ is the design variable which includes the number of reinforcing families (N) (if it is not fixed), the fiber radius (R), the distance between the fibers in the reinforcing layer (δ_f), the distance between the layers of reinforcing fibers (δ_b), the angle between the axes of fibers of the β-family and the Ox_1 axis (φ_β) (see Fig. 1); a_β, b_{ijkl}, and c_{ijkl} are the functions given in [1, 2], b_{ijkl} and c_{ijkl} depending only on the angle of fiber stacking in a given layer, whereas a_β depends only on the remoteness of the βth fiber layer from the Ox_1x_2 plane.

Local Stresses and Fiber Strains. The following formulas for calculation of the axial stresses in fibers σ_f (the remaining stresses in fibers are zero [1]) were derived in [1–5]:

$$\sigma_f = E_f \sum_{i,j=1,2} l_i^\beta l_j^\beta (e_{ij} + x_3 \rho_{ij}) \tag{1.2}$$

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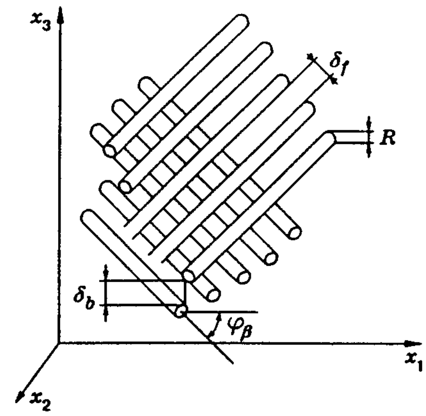

Fig. 1

in the βth layer of reinforcing fibers. Here e_{ij} are the tensile, compressive, and shear strains in the plate plane, ρ_{ij} are the bending and torsional strains which are calculated using the solution of the plate deformation problem with the elastic constants (1.1) according to standard formulas [6], E_f is the Young's modulus of the fiber material, and $\mathbf{l}^\beta$ is the directing vector of the βth family of fibers.

Local Stresses and Strains of the Matrix. The general formulas for local displacements of the fiber skeleton are given in [1, 2]. They allow one to analyze local displacements of a set of fibers as a result of their expansion into interfiber displacements caused by a mutual displacement of the fiber within one reinforcing layer and into interlayer displacements caused by mutual displacements of the neighboring layers of reinforcing fibers.

The result of the analysis performed in [1, 2] is given in Table 1, in which the estimates of microscopic strains of the matrix e_b^γ, which correspond to the averaged strains e_{ij} and curvatures ρ_{ij} (γ in Table 1 is the strain-mechanism number) are presented.

The estimates in Table 1 are more exact compared with those given in [1, 3] (Table 1 does not contain terms with ∇w). The presence of the ∇w-containing terms (overestimating the result) in [1, 3] is caused by the fact that the global bending strains of a plate were not taken into account in [1, 3]. One can derive these estimates from [1, 3] if one takes into consideration that the deflection of the plate as a rigid body does not change strains and, at the same time, make ∇w vanish at the point considered if this point is chosen properly.

As follows from Table 1, to plate macrostrains corresponds the strain spectrum of the matrix at the local level. These local strains are associated with the local structure of the plate.

2. Averaged Strength Criterion. Let us take the local strength criterion of the fiber in the form

$$|\sigma_f| \leqslant \sigma_f^*, \tag{2.1}$$

where σ_f^* is the strength of the fiber.

Substituting formula (1.2) into (2.1), we obtain

$$f_\beta \equiv \Big|E_f \sum_{i,j=1,2} l_i^\beta l_j^\beta (e_{ij} + x_3\rho_{ij})\Big| \Big/ \sigma_f^* \leqslant 1 \tag{2.2}$$

in the βth layer of reinforcing fibers.

For the matrix, there are only the strain estimates. In view of this, we use the nonfailure criterion in the following form:

If the condition

$$|e_b^\gamma| \leqslant e_b^{\gamma *} \tag{2.3}$$

is satisfied for e_b^γ (the upper estimate of the strain), where $e_b^{\gamma *}$ is the ultimate strain of the matrix, which corresponds to the γth mechanism of matrix deformation, the matrix does not fail.

Substitution of the estimate of e_b^γ from Table 1 into (2.3) yields

$$g_\gamma = |e_b^\gamma| / e_b^{\gamma *} \leqslant 1, \tag{2.4}$$

TABLE 1

γ	Strain mechanism	Estimate
1	Interfiber tensions	$\lvert x_3\rvert\,\lvert\rho_{ij}\rvert + \lvert e_{ij}\rvert$
2	Interfiber shears	$\left(\varepsilon\lvert x_3\rvert \dfrac{R}{\delta_f} + \lvert x_3\rvert\right)\lvert\rho_{ij}\rvert + \lvert e_{ij}\rvert$
3	Interfiber torsions	$\left(\varepsilon\lvert x_3\rvert \dfrac{R}{\delta_f} + \lvert x_3\rvert\right)\lvert\rho_{ij}\rvert + \lvert e_{ij}\rvert$
4	Interlayer tensions	$\left(\dfrac{\lvert x_3\rvert}{\delta_b} + \lvert x_3\rvert\right)\lvert\rho_{ij}\rvert$
5	Interlayer torsions	$\left(\varepsilon R + \dfrac{R\lvert x_3\rvert}{\delta_b} + \dfrac{\varepsilon\lvert x_3\rvert R}{\delta_b}\right)\lvert\rho_{ij}\rvert + \dfrac{R}{\delta_b}\lvert e_{ij}\rvert$

where e_b^γ is the estimate which corresponds to the γth mechanism of local strain of the matrix.

Satisfaction of the inequality

$$S(\mathrm{u}, e_{ij}, \rho_{ij}) \equiv \max\{f_1, \ldots, f_N, g_1, \ldots, g_5\} \leqslant 1 \tag{2.5}$$

guarantees the nonfailure of all components of the composite. Criterion (2.5), which contains the global (average) strains e_{ij} and the curvatures ρ_{ij} of the plate, is called an averaged criterion [4, 7–10].

3. Design of a Plate with Given Strain-Strength Characteristics. Formulation of the Problem: it is necessary to reveal whether it is possible to create a plate with a required set of rigidity and strength characteristics using a given set of fibers. If the answer is positive, one should find the solution — design of a composite with prescribed characteristics.

Formalization of the Problem: it is necessary to solve the equation

$$S^{\nu}_{ijkl}(\mathrm{u}) = S^{\nu}_{ijkl} \qquad [(\nu, i, j, k, l) \in \Lambda] \tag{3.1}$$

(Λ is the set of given rigidities; as a rule, not all rigidities are given) with the following limitations on the strength

$$S(\mathrm{u}, e_{ij}, \rho_{ij}) \leqslant 1 \tag{3.2}$$

and on the design variables

$$\min_i \leqslant u_i \leqslant \max_i. \tag{3.3}$$

Here S^{ν}_{ijkl} are the given rigidities and e_{ij} and ρ_{ij} are the given strains and curvatures which should be sustained by the plate. The solution is performed relative to the vector $\mathbf{u} = (R, \delta_f, \delta_b, \varphi_1, \ldots, \varphi_N)$ (with a given number of reinforcing families N).

One possible method of solving problem (3.1)–(3.3) is its reduction to the minimization problem. Let us write the function

$$F(\mathrm{u}) = \sum_{\substack{i,j,k,l=1,2\\ \nu=0,1,2}} A^{\nu}_{ijkl} \frac{|S^{\nu}_{ijkl}(u) - S^{\nu}_{ijkl}|}{S^{\nu *}_{ijkl}} + P(\mathrm{u}), \tag{3.4}$$

where $A^{\nu}_{ijkl} = 1$ if the rigidity S^{ν}_{ijkl} is among the given ones, otherwise $A^{\nu}_{ijkl} = 0$; $P(\mathrm{u})$ is the penalty for violation of conditions (3.2): $P(\mathrm{u}) = 0$ if condition (3.2) is satisfied for $\mathbf{u}$, $P(\mathrm{u}) = \mathrm{const} > 0$ if it is not satisfied; and $S^{\nu *}_{ijkl}$ are the characteristic values of the rigidities. It is evident that $\mathbf{u}$ is the solution of problem (3.1)–(3.3) if $F(\mathrm{u}) = 0$. Since $F(\mathrm{u}) \geqslant 0$ by definition, the problem is reduced to the search for the global minimum of the function (3.4).

4. Algorithms and Software. As algorithms of search for the global minimum of the function (3.4), we used the following ones:

(a) the algorithm of a random search, which was proposed by Annin and Kolpakov [9];

TABLE 2

Problem	Time and accuracy of the solution	
	Algorithm (a)	Algorithm (b)
No. 1	2 min, 1%	9 min, 3%
No. 2	2 min, 1%	3.5 min, 3%

(b) the classical algorithm of gradient drop with a random choice of the initial point (used as a test one).

Algorithm a turned out to be convenient from the viewpoint of visual control of the minimization process. Table 2 gives an idea of the high-speed response of the algorithms [an IBM 486 DX, 66 MHz with an arithmetic processor was employed, and, as the time solution, we took the duration of the procedure for realization of algorithm (a) or (b), respectively].

We chose the following problems as test problems:

Problem No. 1. Fiber, $E_f = 1 \cdot 10^{11}$ Pa, $\nu = 0.2$, the number of reinforcing fibers is $N = 4$, and $e_{ij} = \rho_{ij} = 0$ (design without limitations on strength properties). It is required to design a composite with $S^0_{1111} = 1 \cdot 10^7$ and $S^0_{2222} = 1 \cdot 10^7$ (rigidities in the plate plane) and with $S^2_{1111} = 1 \cdot 10^{-1}$ and $S^2_{2222} = 2 \cdot 10^{-1}$ (flexural rigidities).

Problem No. 2. Fiber, the restrictions and the rigidities are the same as in problem No. 1; $e_{11} = 0.02$ and $e_{12} = e_{22} = 0$ and $\rho_{11} = 0.02$ and $\rho_{12} = \rho_{22} = 0$. The ultimate deformation of the fibers is $e^*_f = 0.02$, and that of the matrix is $e^{\gamma *}_f = 0.05$. The design is performed with allowance for strength properties.

5. Estimation of Parameters in the Design. In solving the design problem, the difficulty arises of how to predict the possible values of the rigidities $S^{\nu}_{ijkl}(\mathbf{u})$. The above procedure is oriented to a designer who knows exactly which characteristics are required, and will be satisfied if the designer obtains the solution, if it exists, or if the designer concludes that there is no solution. In practice, as early as the first stage of design, it is desirable to get an idea of the possible values of the quantities to draw a final conclusion on their values at the second stage.

The algorithms and programs used can be improved with a view to obtaining those that allow one to estimate the interval of possible values of the quantity considered, with other quantities specified. In the above consideration, we can use other similar functions [for example, the margin of the strength criterion $S(\mathbf{u})$ and the volume content of fiber] as specified characteristics. The margin of the strength criterion is determined by condition (2.5) and can be included in the number of specified (and, in practice, this is more interesting than the estimated ones, see Sec. 4) characteristics along with the rigidities.

In practice, the volume content of fibers $\gamma(\mathbf{u})$ is of interest owing to the fact that it determines to a considerable extent the weight and cost characteristics of the composite.

6. Examples. EXAMPLE 1. Let us take $E_f = 1 \cdot 10^{11}$ Pa, $\nu = 0.2$, $\sigma^*_f = 0.02 \cdot 10^{11}$ Pa, and $e^{\gamma *}_b = 0.02$; the constraints on the geometrical sizes are as follows: $0 < R < 1$ m^{-4}, $0 < \delta_f < 1$ m^{-4}, $0 < \delta_b < 1$ m^{-4}, and $0 < \varphi_\beta < 3.14$ rad; the number of reinforcing layers is $N = 4$; and the strains are as follows: $e_{11} = 0.01$, $e_{12} = e_{22} = 0$, and $\rho_{ij} = 0$. The design is performed taking into account the strength criterion.

In what follows, we present the protocol of the solution.

1. We set $S^0_{1111} = 1 \cdot 10^7$ and estimate S^0_{2222}; we obtain $0 < S^0_{2222} < 11.058 \cdot 10^7$.

2. We use $S^0_{2222} = 5 \cdot 10^7$ and estimate the flexural rigidity S^2_{1111}; we have $2.845 < S^2_{1111} < 24.241 \cdot 10^{-1}$.

3. We use $S^2_{1111} = 15 \cdot 10^{-1}$ and estimate the flexural rigidity S^2_{2222}; we obtain $15.311 < S^2_{2222} < 51.767 \cdot 10$.

4. We use $S^2_{2222} = 20 \cdot 10^{-1}$ and estimate the margin of the strength criterion. For it, we have the coincidence of the minimum and maximum values. Both are equal to 0.5.

5. Let us solve the problem with $S^0_{1111} = 1 \cdot 10^7$, $S^0_{2222} = 5 \cdot 10^7$, $S^2_{1111} = 15 \cdot 10^{-1}$, and $S^2_{2222} = 20 \cdot 10^{-1}$, without fixation of the margin of strength. We obtain the following parameters: $R = 0.69$ m^{-4}, $\delta_f = 0.94$ m^{-4},

TABLE 3

S^0_{1111}	S^0_{2222}	S^0_{1212}	S^0_{1112}	S^0_{1222}	S^2_{1111}	S^2_{2222}	S^2_{1212}	S^2_{1112}
0	0	0	−4.71	−4.38	0	0	0	−238.66
13.61	14.36	3.43	4.53	4.76	145.82	113.66	38.73	239.29

and $\delta_b = 0.38\ \text{m}^{-4}$, the angles of fiber stacking are 1.20, 2.01, 1.76, and 0.62 rad.

The local margins of the strength criterion for this project are as follows: in the fibers, $f_1 = 0.07$, $f_2 = 0.09$, $f_3 = 0.02$, and $f_4 = 0.33$; the interfiber tensile strains in the matrix (see Table 1) are equal to 0.50, the interfiber shear and torsion strains are 0.50, the interlayer tensile strains are 0.50; and the interlayer torsion strains are 0.50.

EXAMPLE 2. We set $E_f = 1 \cdot 10^{11}$ Pa and $\nu = 0.2$; the number of reinforcing layers is $N = 6$; the limitations on the sizes are as follows: $0 < R < 1\ \text{m}^{-4}$, $0 < \delta_f < 1\ \text{m}^{-4}$, $0 < \delta_b < 1.0\ \text{m}^{-4}$, and $0 < \varphi_\beta < 3.14$ rad; the strains are as follows: $e_{ij} = \rho_{ij} = 0$. The strength is not taken into account.

1. We set $S^0_{1111} = S^0_{2222} = 1 \cdot 10^7$ and estimate the volumetric content of the fiber; we obtain the estimate $0.055 < \gamma < 0.342$.

2. We solve the problem with $S^0_{1111} = S^0_{2222} = 1 \cdot 10^7$ and $\gamma = 0.06$, which is close to the minimum value of γ. We obtain the following parameters: $R = 1.00\ \text{m}^{-4}$, $\delta_f = 1.00\ \text{m}^{-4}$, and $\delta_b = 0.13\ \text{m}^{-4}$, and the angles of fiber stacking are as follows: 1.48, 3.03, 0.20, 1.72, 3.09, and 1.41 rad.

EXAMPLE 3. Calculation of the estimates of the possible values of the plate's rigidities which can be obtained using fibers with $E_f = 1 \cdot 10^{11}$ Pa, $\nu = 0.2$, $N = 4$, and $e_{ij} = \rho_{ij} = 0$ (there are no strength limitations); the restrictions on the geometrical sizes are as follows: $0 < R < 1\ \text{m}^{-4}$, $0 < \delta_f < 1\ \text{m}^{-4}$, $0 < \delta_b < 1\ \text{m}^{-4}$, and $0 < \varphi_\beta < 3.14$ rad; the restrictions on the other parameters are absent.

The results are given in Table 3, where the minimum and maximum rigidity (rigidities in the plane and flexural ones) values are given.

Table 3 also enables one to evaluate the accuracy of the estimates obtained. Owing to the symmetry of the problem, the exact estimates of S^ν_{1111} and S^ν_{2222} must coincide, and the intervals of the S^ν_{1112} and S^ν_{1222} values must be symmetric.

7. Thermoelasticity of Reinforced Plates. Let the fibers have the coefficient of thermal expansion α_f and the matrix α_b. For the thermoelasticity problem, it is possible to perform an analysis similar to that for the elastic problem. This analysis is based on the analog of formula (1.3) which relates the local displacements $\mathbf{u}^\varepsilon$ to the temperature θ [1]:

$$\mathbf{u}^\varepsilon = \mathbf{S}(\mathbf{x}/\varepsilon)\theta(x_1, x_2), \tag{7.1}$$

where $\mathbf{S}$ is the solution of the cellular problem of thermoelasticity for plates.

For loose frames of fibers, the solution of the cellular problem of thermoelasticity is given in [1]. It corresponds to the thermal expansion of each fiber. Following from this fact, we obtain the following formulas and estimates.

The thermal-expansion constants in the plate plane S^0_{ij} and the flexural ones S^1_{ij} are given by the formulas

$$S^0_{ij}(\mathbf{u}) = \alpha_f E_f \sum_{\beta=1}^{N} s_{ij}(\varphi_\beta)\gamma, \quad S^1_{ij}(\mathbf{u}) = \alpha_f E_f \sum_{\beta=1}^{N} s_{ij}(\varphi_\beta) a_\beta \gamma, \tag{7.2}$$

where $i, j = 1$ and 2, the function s_{ij} is given in [1]. The first formula in (7.2) holds true if $S^0_{ii}(\mathbf{u}) \neq 0$, and if $S^0_{ii}(\mathbf{u}) = 0$, one should set $S^0_{ii}(\mathbf{u}) = \alpha_b E_b$ (E_b is the Young's modulus of the matrix). The latter case corresponds to the absence of reinforcement in the direction of the Ox_i axis; in practice, this does not occur, but it should be taken into consideration in solving the problem.

By thermal-expansion constants, we mean the coefficients at θ in the governing equations. The coefficients of thermal expansion of the plate are $\{S^0_{ijkl}\}^{-1}\{S^0_{kl}\}$ and $\{S^2_{ijkl}\}^{-1}\{S^1_{kl}\}$ [11, 12].

Local Stresses in Fibers. The local stresses in fibers are calculated using the replacement of e_{ij} by $e_{ij} - \alpha_f \theta \delta_{ij}$ in (1.2).

Local Stresses in the Matrix. For loose frames of fibers, formula (7.1) describes its increase in the scale $1 : 1 + \alpha_f \theta$. This uniform increase does not lead to a mutual displacement of the frame points. Like the frame, the matrix undergoes a strain estimated as $\alpha_f \theta$. The natural thermal deformations of the matrix are estimated as $\alpha_b \theta$. As a result, incompatibility in the thermal strains of the fiber and of the matrix, which is estimated as $(\alpha_f - \alpha_b)\theta$, arises, and the stresses which are caused by this incompatibility are estimated as $E_b(\alpha_f - \alpha_b)\theta$.

One should bear in mind that the mechanism described above does not work in the nonreinforced directions (for example, across the plate). In these directions, the fibers and the matrix expand freely, and this free expansion yields zero stresses having no effect on the strength. In this connection, this case may be omitted.

The formulation of the design problem with allowance for thermoelastic characteristics and stresses and the methods of its solution are not different from those presented in this study.

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